Laws and Theorems of Boolean Logic

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Axioms of Boolean Algebra (1 of 4)

- Axiom 0
  - Set of elements, B
  - Two binary operators, + and ·
  - One unary operator, '

- Axiom 1
  - Set B contains at least two elements a and b s.t. $a \neq b$

Axioms of Boolean Algebra (2 of 4)

- Axiom 2 – *Closure*
  - For every a and b in B,
    - a + b is in B
    - $a \cdot b$ is in B
- Axiom 3 – *Commutative laws*
  - For every a and b in B,
    - a + b = b + a
    - $a \cdot b = b \cdot a$

Axioms of Boolean Algebra (3 of 4)

- Axiom 4 – *Associative laws*
  - For every a, b, and c in B,
    - $(a + b) + c = a + (b + c) = a + b + c$
    - $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$
- Axiom 5 – *Identities*
  - There exists an identity element with respect to +, designated by 0, s.t.
    - $a + 0 = a$, for every a in B
  - There exists an identity element with respect to ·, designated by 1, s.t.
    - $a \cdot 1 = a$, for every a in B

Axioms of Boolean Algebra (4 of 4)

- Axiom 6 – *Distributive laws*
  - For every a, b, and c in B,
    - $a + (b \cdot c) = (a + b) \cdot (a + c)$
    - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

- Axiom 7 – *Complement*
  - For each a in B, there exists an element a' in B (the complement of a) s.t.
    - $a + a' = 1$
    - $a \cdot a' = 0$

Order of Evaluation of Boolean Expressions

- In a Boolean expression without parentheses,
  - Complement is always applied first
  - then, AND is applied
  - finally, OR is applied

- Operations in sub-expressions within parentheses are always applied before operations between parenthesized expressions

- Equivalently, we say that parentheses have highest precedence, followed by complementation, followed by AND, then finally by OR

Literals and Simplification

- In a Boolean expression, each variable's appearance in either its non-complemented or complemented form is called a *literal.* A literal represents the connection of a variable or its complement to a unique gate input. Do *not* include the output variable.
- If E_1 and E_2 are two expressions for the same Boolean function (*i.e.,* they have the same truth table), we say that E_2 is simpler than E_1 if it contains fewer literals

- For example, the expression:

$$Sum = \bar{A}\bar{B}Carry_{in} + \bar{A}B\overline{Carry_{in}} + A\bar{B}\overline{Carry_{in}} + ABCarry_{in}$$

contains 3 variables (A, B, and $Carry_{in}$) and 12 literals

Duality

- Every expression in Boolean logic has a **dual expression** that is formed by
  - Replacing every AND operator by an OR operator, and vice versa
  - Replacing every constant 0 by 1, and vice versa

  - Be sure to not change the order of operations when replacing AND by OR and vice versa (*i.e.,* assume every operation is parenthesized)

- For every ***statement*** of Boolean logic that is true, its *dual* is also true

- For example, $x + 0 = x$ is true
- $(x + 0 = x)^D$ which is $x \cdot 1 = x$ is also true

- In the following laws and theorems, we will present the *dual* with a D suffix

Observation on Duality

- Does *duality* remind you of *active-high* vs. *active-low* logic?
  - It should!

Other Theorems That Can be Derived from the Laws and Theorem Above

- All of the Theorems that follow can be derived from the Axioms, Laws, and Theorems given above

Operations with 0 and 1

- 1. $x + 0 = x$
- 1D. $x \cdot 1 = x$

- 2. $x + 1 = 1$
- 2D. $x \cdot 0 = 0$

Idempotent Theorem

- 3. $x + x = x$
- 3D. $x \cdot x = x$

Involution Theorem

- 4. $(x')' = x$

Theorem of Complementarity

- 5. $x + x' = 1$
- 5D. $x \cdot x' = 0$

Commutative Law

- 6. $x + y = y + x$
- 6D. $x \cdot y = y \cdot x$

Associative Law

- 7. $(x + y) + z = x + (y + z) = x + y + z$
- 7D. $(x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot y \cdot z$

Distributive Law

- 8. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- 8D. $x + (y \cdot z) = (x + y) \cdot (x + z)$

Simplification Theorems

- 9. $x \cdot y + x \cdot y' = x$
- 9D. $(x + y) \cdot (x + y') = x$
  - Notice the insertion of parentheses to ensure that the order of operations has not changed

- 10. $x + x \cdot y = x$
- 10D. $x \cdot (x + y) = x$
  - Once again, notice the insertion of parentheses to ensure that the order of operations has not changed

- 11. $(x + y') \cdot y = x \cdot y$
- 11D. $x \cdot y' + y = x + y$

DeMorgan's Law

- 12. $(x + y + z + \ldots)' = x' \cdot y' \cdot z' \cdot \ldots$
- 12D. $(x \cdot y \cdot z \cdot \ldots)' = x' + y' + z' + \ldots$

General Form of DeMorgan's Law

- 13. $\{f(x_1, x_2, \ldots, x_n, 0, 1, +, \cdot)\}' = f(x_1', x_2', \ldots, x_n', 1, 0, \cdot, +)$

Duality

- 14. $(x + y + z + \ldots)^D = x \cdot y \cdot z \cdot \ldots$
- 14D. $(x \cdot y \cdot z \cdot \ldots)^D = x + y + z + \ldots$

General Form of Duality

- 15. $\{f(x_1, x_2, \ldots, x_n, 0, 1, +, \cdot)\}^D = f(x_1, x_2, \ldots, x_n, 1, 0, \cdot, +)$

Theorem for Multiplying and Factoring

- 16. $(x + y) \cdot (x' + z) = x \cdot z + x' \cdot y$
- 16D. $x \cdot y + x' \cdot z = (x + z) \cdot (x' + y)$

Consensus Theorem

- 17. $x \cdot y + y \cdot z + x' \cdot z = x \cdot y + x' \cdot z$
- 17D. $(x + y) \cdot (y + z) \cdot (x' + z) = (x + y) \cdot (x' + z)$